COMBINATORICA 6 (4) (1986) 315—320

COUNTEREXAMPLES TO CONJECTURES ON 4-CONNECTED MATROIDS

COLLETTE R. COULLARD

Received 20 May 1985
Revised 30 October 1985

Tutte characterized binary matroids to be those matroids without a U_4^2 minor. Bixby strengthened Tutte's result, proving that each element of a 2-connected non-binary matroid is in some U_4^2 minor. Seymour proved that each pair of elements in a 3-connected non-binary matroid is in some U_4^2 minor and conjectured that each triple of elements in a 4-connected non-binary matroid is in some U_4^2 minor. A related conjecture of Robertson is that each triple of elements in a 4-connected non-graphic matroid is in some circuit. This paper provides counterexamples to these two conjectures.

1. Introduction

Familiarity with matroid theory is assumed. For an introduction, see Welsh [11].

Let M be a matroid on E with rank function r. M^* denotes the dual of M, with rank function $r^*(A)=|A|-r(E)+r(E-A)$, for $A\subseteq E$, where $|A|$ is the cardinality of A. A *loop* of M is a one-element circuit, and a *coloop* is a one-element cocircuit. Distinct elements $e,f\in E$ are *parallel* if $\{e,f\}$ is a circuit and *in series* is $\{e, f\}$ is a cocircuit. A *triangle* is a 3-element circuit and a *triad* is a 3-element cocircuit.

For $X\subseteq E$, $M\setminus X$ denotes the deletion of X and M/X the contraction of X. For X, Y disjoint subsets of E, $N=M\setminus X/Y=M/Y\setminus X$ is a *minor* of M. $E(N)$ denotes the set of elements of N. We say that N *contains* set A if $A\subseteq E(N)$.

Given integers $0\leqq n\leqq m\neq 0$, U_m^n denotes the *uniform* matroid on m elements, in which every n-element subset is a base. For a graph G, $\mathcal{M}(G)$ denotes the usual polygon matroid of G. For integer $n\geqq 3$, $\mathcal{W}_n$ denotes the whirl matroid with n spoke elements and n rim elements.

Let M be a matroid on E. A partition $\{S, T\}$ of E is called a (Tutte) *k-separation* [10], for some positive integer k, if $|S|\geqq k\leqq |T|$ and $r(S)+r(T)-r(E)\leqq \leqq k-1$, or, equivalently, $r(S)+r^*(S)-|S|\leqq k-1$. M is *n-connected*, for some integer $n\geqq 2$, if M has no k-separation for $k<n$.

A matroid is *binary* if it can be represented over GF(2). In [9] Tutte characterized binary matroids to be those matroids without a U_4^2 minor. Bixby strengthened Tutte's result in [1], proving that a 2-connected matroid is non-binary if and

AMS subject classification (1980): 05 B 35 only if each element is in some U_4^2 minor. In [6] Seymour provided the analog of Bixby's result for 3-connected matroids; he proved that a 3-connected matroid is non-binary if and only if every pair of elements is in some U_4^2 minor. (Later in [7] he showed how that result follows directly from his theory of 2-roundedness.)

In [6], Seymour made the following natural conjecture for 4-connected matroids.

Conjecture 1. [6] *If M is a 4-connected non-binary matroid and e, f and g are three elements of M, then M has a U_4^2 minor containing $\{e, f, g\}$.*

Oxley [3] proved that every triple in a 3-connected non-binary matroid is either contained in some U_4^2 minor or is the set of spoke or rim elements of some $\mathcal{W}_3$ minor. Oxley's result provided some evidence that Conjecture 1 was indeed true.

A necessary condition for a set of elements to be contained in a U_4^2 minor is that it be contained in both a circuit and a cocircuit. Thus, the following conjecture of Robertson, when restricted to non-binary matroids, is a weakening of Conjecture 1.

Conjecture 2. [6] *If a, b, c are three elements of a 4-connected non-graphic matroid, then there is a circuit containing $\{a, b, c\}$.*

In [4], Robertson and Chakravarti proved Conjecture 2 for cographic matroids. Seymour [5] has proved that every 4-connected binary matroid is either graphic, cographic, or R_{10}, the 10-element matroid represented over GF(2) by the matrix of all possible vectors consisting of three ones and two zeros. Having verified that Conjecture 2 holds for R_{10}, Seymour's theorem implies Conjecture 2 is true for regular matroids. In [8], Seymour generalized this result to binary matroids, leaving the more general non-binary problem still open.

In Section 2, we present a 12-element counterexample to Conjecture 1 and prove that this counterexample is minimal. In Section 3, we present a counterexample to Conjecture 2 for non-binary matroids.

2. Counterexample to Conjecture 1

Let P_{12} be the 12-element matroid represented over **R**, the real numbers, by matrix A_{12} below, where the element set of P_{12} is $\{a, \ldots, l\}$, as labeled. The matrix is partitioned to display its symmetry and structure. The non-identity part is symmetric. Notice the four obvious $\mathcal{W}_3$ minors.

$$
\begin{array}{cccccccccccc}
a & b & c & d & e & f & g & h & i & j & k & l
\end{array}
$$
$$
\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}\right]
$$

Fig. 1. Matrix A_{12}

Lemma 1. P_{12} *is non-binary.*

Proof. The lemma follows from Tutte's characterization of binary matroids and the fact that $U_4^2 \cong P_{12} \backslash \{c, j, k, l\} / \{d, e, f, g\}$. ∎

Lemma 2. P_{12} *has no* U_4^2 *minor contaning* $\{a, b, c\}$.

Proof. Suppose $N = P_{12} \backslash X / Y \cong U_4^2$, where X is coindependent and Y is independent, with $\{a, b, c\} \subseteq E(N)$. Since $r(P_{12}) = r^*(P_{12}) = 6$ and $r(N) = r^*(N) = 2$, $|X| = |Y| = 4$. It is easy to show that $\{a, b, d, e\}$, $\{a, c, d, f\}$, and $\{b, c, e, f\}$ are cocircuits of P_{12}; thus $|\{d, e, f\} \cap X| \leqq 1$.

For any $y \in \{g, h, i, j, k, l\}$, $\{a, b, c\}$ contains a loop or parallel elements in $P_{12}/\{d, e, f, y\}$. Thus $\{d, e, f\} \nsubseteq Y$. Suppose $|\{d, e, f\} \cap Y| = 2$. By symmetry, assume $\{e, f\} \subseteq Y$. Then since a and j are parallel in $P_{12}/\{e, f\}$, it must be that $j \in X$. Since $\{a, h, i\}$ is a triad of $P_{12} \backslash j$, either $h \in Y$ or $i \in Y$, yielding a contradiction in either case, due to circuits $\{a, c, e, h\}$ and $\{a, b, f, i\}$ of P_{12}. Thus, $|\{d, e, f\} \cap Y| = 1$, and $|\{d, e, f\} \cap X| = 1$.

By symmetry, assume $E(N) = \{a, b, c, d\}$, $e \in X$ and $f \in Y$. Since $\{a, b, i\}$ is a circuit of P_{12}/f, it must be that $i \in X$. Similarly, $\{b, d, l\}$ is a triad of $P_{12} \backslash i$, implying that $l \in Y$. Now $\{a, d, h\}$ and $\{b, d, k\}$ are triangles of $P_{12}/\{f, l\}$, implying that $X = \{e, h, i, k\}$, and hence $Y = \{f, g, j, l\}$. But $\{a, b, f, g, j, l\}$ is a circuit of P_{12}, implying that a and b are parallel in N, a contradiction. ∎

We show that P_{12} is 4-connected indirectly, making use of the following observation.

Proposition 1. *Let* A *be a* $(0, 1)$*-matrix and let* M_1, M_2 *be the matroids on* E, *the column set of* A, *represented by* A *over* $\mathbf{R}$ *and* GF(2), *respectively. Assume* M_1 *and* M_2 *have the same rank. Every k-separation of* M_1 *is a k-separation of* M_2.

Proof. For any square submatrix D of A, the determinant of D can be written as a sum of $+1$'s and -1's (where $+1 = -1$ over GF(2)). If the determinant over $\mathbf{R}$ is zero, then there is an even number of terms in the sum, implying that the determinant over GF(2) is zero. It follows that $r_2 \leqq r_1$, where r_i is the rank function of M_i, $i = 1, 2$, which, together with the fact that M_1 and M_2 have the same rank, implies that if $\{S, T\}$ is a k-separation of M_1, then $\{S, T\}$ is a k-separation of M_2. ∎

Let B_{12} be the matroid represented by A_{12} over GF(2). By Proposition 1, if B_{12} is 4-connected, then P_{12} is 4-connected.

Lemma 3. B_{12} *is* 4-*connected.*

Proof. It is easy to check that B_{12} has no loops, coloops, series or parallel elements, triangles or triads. Suppose $\{S, T\}$ is a k-separation of B_{12}, $k \leqq 3$, with $|S| \leqq |T|$. Then $|S| \geqq 4$. We claim that B_{12} has no 4-element circuit that is also a cocircuit. Suppose such a circuit C exists. Let B be the base $\{a, b, c, d, e, f\}$ of B_{12}, and B^* the cobase $\{g, h, i, j, k, l\}$. If $|C \cap B| = 3$, then C is a fundamental circuit with respect to B. It is easy to check that no fundamental circuit with respect to B is a cocircuit. Similarly, $|C \cap B^*| \neq 3$. Thus, $|C \cap B| = |C \cap B^*| = 2$. The circuits of this type can be enumerated by finding all pairs of columns of B^* that meet in two rows.

These circuits are: $\{c, f, g, k\}$, $\{b, e, g, l\}$, $\{c, f, h, j\}$, $\{a, d, h, l\}$, $\{b, e, i, j\}$, $\{a, d, i, k\}$, none of which is a cocircuit. Having verified the claim, we may assume $|S| \geqq 5$.

Suppose $|S|=5$. Then $r(S)+r^*(S) \leqq 7$, from which it follows that $r(S) \leqq 3$ or $r^*(S) \leqq 3$. Since $B_{12} \cong B_{12}^*$, the dual of B_{12}, assume $r(S) \leqq 3$. Since there are no circuits with fewer than 4 elements, $r(S)=3$ and S is the union of 4-element circuits. But then S contains two distinct circuits C_1, C_2 with $|C_1 \cap C_2|=3$, implying, since B_{12} is binary, that there are parallel elements, a contradiction.

Assume $|S|=|T|=6$. Then $r(S)+r^*(S) \leqq 8$. Again $r(S) \leqq 3$ implies a contradiction. We conclude that $r(S)=r^*(S)=4$. If S contains a 5-element circuit C_1, then S contains a circuit C_2 such that $|C_1 \cap C_2| \geqq 3$, implying that B_{12} has a triangle or parallel elements, a contradiction. Assume, therefore, that S is the union of 4-element circuits that pairwise have intersection of size 2. Similarly, S is the union of 4-element cocircuits. Since no 4-element circuit is a cocircuit, S contains a 4-element circuit C and a 4-element cocircuit C^* such that $|C \cap C^*|=3$, contradicting the fact that B_{12} is binary. ▮

Lemmas 1, 2 and 3, together with Proposition 1, establish that P_{12} is indeed a counterexample to Conjecture 1. Next we show that P_{12} is a minimal counterexample.

Lemma 4. *P_{12} has no proper minor that provides a counterexample to Conjecture* 1.

Proof. Suppose M is a non-binary proper minor of P_{12} and has 3 elements x, y, and z not contained in a U_4^2 minor. Let X, Y be coindependent, independent, respectively, such that $M=P_{12} \setminus X/Y$. By duality, assume $X \neq \emptyset$. Let $B=\{a, b, c, d, e, f\}$ and $B^*=\{g, h, i, j, k, l\}$. Suppose $|X \cap B^*| \neq \emptyset$. By symmetry, assume $g \in X$. Now $P_{12} \setminus g$ has triads $\{b, i, k\}$, $\{c, h, l\}$, $\{d, k, l\}$, $\{b, f, j\}$, $\{c, e, j\}$. Since M is 4-connected, $T \cap Y \neq \emptyset$, for each triad T in this list. It follows that $|Y| \geqq 3$. Similarly, if $X \cap B \neq \emptyset$, then $|Y| \geqq 3$. Thus, we can assume $|Y| \geqq 3$. The same argument provides $|X| \geqq 3$. Clearly $|Y|=3$ or $|X|=3$. By duality, assume $|X|=3$. Then $|Y|=3$ or 4. If $|Y|=4$, then M is a rank 2, 1-element extension of U_4^2. It is easy to check that the only 3-connected rank 2, 1-element extension of U_4^2 is U_5^2, a matroid that is not a counterexample. Thus $|Y|=3$. It follows that M is a rank 3, 2-element extension of U_4^2. Since M is 4-connected, M has no circuits or cocircuits with fewer than 4 elements. Thus, $M \cong U_6^3$, and again is not a counterexample. This completes the proof. ▮

We have recently learned that Jeff Kahn [2] has also found a counterexample to Conjecture 1. We note that Kahn's example, call it M_K, has 15 elements and is not shown to be minimal. As Kahn states in his paper, M_K is obtained from a binary matroid by declaring one circuit/hyperplane to be independent. (This construction is known to produce a matroid.) He asks if every counterexample to Conjecture 1 can be obtained in this way. The answer is no. To see that P_{12} is not the result of this construction, observe that if it were, then B_{12} would be the original binary matroid, since base $B=\{a, b, c, d, e, f\}$, having fundamental circuits with fewer than seven elements, could not be the new base. Now B_{12} has 2 circuit/hyperplanes that are independent in P_{12}: $C_1=\{a, b, c, j, k, l\}$ and $C_2=\{d, e, f, g, h, i\}$. One can show that P_{12} is obtained from B_{12} by first declaring C_1 to be independent and then declaring C_2 to be independent.

3. Counterexample to Conjecture 2

This section presents a counterexample to Conjecture 2 for non-binary matroids. Unfortunately, P_{12} does not provide a counterexample; each triple is contained in some circuit.

Let P_{16} be the matroid represented over **R** by the matrix D_{16} in Figure 2, where the element set of P_{16} is $\{a, \ldots, p\}$, as labeled. Note that D_{16} is a standard representative matrix for $\mathcal{M}(K_{4,4})$ as labeled in Figure 3. Since $\mathcal{M}(K_{4,4})$ is 4-connected, P_{16} is 4-connected, by Proposition 1.

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
1	0	0	0	0	0	0	1	0	0	1	1	1	1	0	0
0	1	0	0	0	0	0	1	1	0	0	1	0	0	1	1
0	0	1	0	0	0	0	0	1	1	0	0	1	1	0	1
0	0	0	1	0	0	0	0	0	1	1	0	0	0	1	0
0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0
0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1

Fig. 2. Matrix D_{16}

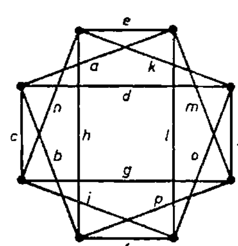

Fig. 3. Graph $K_{4,4}$

Lemma 5. *P_{16} has no circuit that contains $\{a, b, c\}$.*

Proof. Suppose C is a circuit of P_{16} with $\{a, b, c\} \subseteq C$. Then $C - \{a, b, c\}$ is a circuit of $M = P_{16}/\{a, b, c\}$. Now M is graphic, represented by G in Figure 4. For each circuit C_1 of M, there exists a unique circuit C_2 of P_{16} such that $C_1 \subseteq C_2$. Below is a list of all circuits of M. In parentheses after each circuit in the list is the collection of elements from $\{a, b, c\}$ needed to complete that circuit to a circuit of P_{16}. Since $\{a, b, c\}$ does not appear in parentheses, the proof is complete. ▮

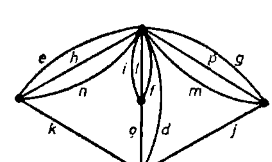

Fig. 4. Graph G

Table 1. Circuits of M

kegj (ac)	*khfo (b)*	*ofd (b)*
kepj (ab)	*khlo (ab)*	*old (a)*
kemj (a)	*khio (bc)*	*oid (c)*
khgj (bc)	*knfo (bc)*	*dgj (c)*
khpj (-)	*knlo (ac)*	*dpj (b)*
khmj (ab)	*knio (-)*	*dmj (a)*
kngj (c)	*ofgj (bc)*	*eh (ab)*
knpj (bc)	*ofpj (b)*	*en (ac)*
knmj (ac)	*ofmj (ab)*	*hn (bc)*
ked (a)	*olgj (ac)*	*il (ac)*
khd (b)	*olpj (ab)*	*if (bc)*
knd (c)	*olmj (-)*	*lf (ab)*
kefo (ab)	*oigj (c)*	*mp (ab)*
kelo (a)	*oipj (bc)*	*mg (ac)*
keio (ac)	*oimj (ac)*	*pg (bc)*

We have not determined whether P_{16} is a minimal counterexample to Conjecture 2. Note that P_{16} is non-binary, and is hence a counterexample to Conjecture 1 as well. Finally, it can be seen that P_{16} is not obtained from a binary matroid by declaring one circuit/hyperplane to be independent.

References

[1] R. E. Bixby, l-matrices and a Characterization of Non-Binary Matroids, *Discrete Math.* **8** (1974), 139—145.
[2] J. Kahn, A problem of P. Seymour on Non-Binary Matroids, *Combinatorica* **5** (1985), 319—323.
[3] J. G. Oxley, On Non-Binary 3-Connected Matroids, *Trans. Amer. Math. Soc., to appear.*
[4] N. Robertson and K. Chakravarti, Covering Three Edges with a Bond in a Nonseparable Graph (abstract), *Annals of Discrete Mathematics* **8** (1980), 247.
[5] P. D. Seymour, Decomposition of Regular Matroids, *Journal of Combinatorial Theory (B)* **28** (1980), 305—359.
[6] P. D. Seymour, On Minors of Non-Binary Matroids, *Combinatorica* **1** (1981), 387—394.
[7] P. D. Seymour, Minors of 3-connected Matroids, *European Journal of Combinatorics* **6** (1985), 375—382.
[8] P. D. Seymour, Triples in Matroid Circuits (1983), *to appear in European Journal of Combinatorics.*
[9] W. T. Tutte, Lectures on Matroids, *Journal of Research of the National Bureau of Standards* **69B** (1965), 1—47.
[10] W. T. Tutte, Connectivity in Matroids, *Canadian Journal of Mathematics* **18** (1966), 1301—1324.
[11] D. J. A. Welsh, *Matroid Theory*, Academic Press, London (1976).

Collette R. Coullard

School of Industrial Engineering
Purdue University
West Lafayette, Indiana 47907
U.S.A.